

DEFORMED NONCOMMUTATIVE TORI

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ABSTRACT. We recall a construction of non-commutative algebras related to a one-parameter family of (deformed) spheres and tori, and show that in the case of tori, the $*$ -algebras can be completed into C^* -algebras isomorphic to the standard non-commutative torus. As the former was constructed in the context of matrix (or fuzzy) geometries, it provides an important link to the framework of non-commutative geometry, and opens up for a concrete way to study deformations of non-commutative tori.

Furthermore, we show how the well-known fuzzy sphere and fuzzy torus can be obtained as formal scaling limits of finite-dimensional representations of the deformed algebras, and their projective modules are described together with connections of constant curvature.

1. INTRODUCTION

In [ABH⁺09a, ABH⁺09b], sequences of matrix algebras were constructed as “fuzzy” analogues of surfaces in \mathbb{R}^3 . In particular, a one parameter family of surfaces, interpolating between spheres and tori, was considered and all finite dimensional (hermitian) representations of the corresponding non-commutative algebras were found and classified. It was shown that the representation theory clearly reflects the topology of the surfaces and that smooth deformations of the geometry induce smooth changes in the representations (see also [Arn08b, AS09]).

One may wonder if there is a relation between the non-commutative algebras constructed in this way and the C^* -algebra representing the standard non-commutative torus [Con80]? Although the construction of the two algebras is completely different (and for the deformed algebras, there is a priori no concept of norm), they correspond to the same (non-commutative) topology. From this point of view a C^* -algebraic isomorphism between the two kind of algebras would be natural.

In this note, we review the construction of the non-commutative algebras related to (deformed) spheres and tori, and show that there exist formal scaling limits in which the representations become the standard fuzzy sphere and torus. Furthermore, we construct a basis in which one can complete the deformed torus algebras into C^* -algebras isomorphic to the standard non-commutative torus. Finally, we consider projective modules of the non-commutative torus in this framework, together with connections of constant curvature.

2. CONSTRUCTION OF NONCOMMUTATIVE ALGEBRAS

Let us briefly recall how to construct non-commutative algebras related to level sets of a polynomial in \mathbb{R}^3 [ABH⁺09b, ABH⁺09a, Arn08b, AS09, Arn08a]. Given a polynomial $C \in \mathbb{R}[x, y, z] \equiv \mathbb{R}[x^1, x^2, x^3]$, one can define a Poisson bracket by

setting

$$(2.1) \quad \{f, g\} = \nabla C \cdot (\nabla f \times \nabla g),$$

for $f, g \in C^\infty(\mathbb{R}^3)$. In particular, it follows that $\{x^i, x^j\} = \varepsilon^{ijk} \partial_k C$. By construction, the polynomial $C(x, y, z)$ Poisson commutes with all functions, which implies that the Poisson structure restricts to the inverse image $C^{-1}(0)$. Thus, it defines a Poisson structure on the quotient algebra $\mathbb{R}[x, y, z] / \langle C(x, y, z) \rangle$, which can be identified with polynomial functions on $C^{-1}(0)$.

To construct a non-commutative version of the above algebra, one starts with the free (non-commutative) associative algebra $\mathbb{C}[X, Y, Z]$ and imposes the relations

$$\begin{aligned} [X, Y] &= i\hbar : \partial_z C : \\ [Y, Z] &= i\hbar : \partial_x C : \\ [Z, X] &= i\hbar : \partial_y C : \end{aligned}$$

where $\hbar \in \mathbb{R}$ and $: \partial_i C :$ denotes a choice of ordering of the (commutative) polynomial $\partial_i C$. In [ABH⁺09a, ABH⁺09b], the authors considered the polynomial

$$(2.2) \quad C(x, y, z) = \frac{1}{2}(x^2 + y^2 - \mu)^2 + \frac{1}{2}z^2 - \frac{1}{2}$$

whose inverse image $C^{-1}(0)$ describes a sphere for $\mu < 1$ and a torus for $\mu > 1$. One computes that

$$\{x, y\} = z \quad \{y, z\} = 2x(x^2 + y^2 - \mu) \quad \{z, x\} = 2y(x^2 + y^2 - \mu)$$

and the corresponding non-commutative relations were chosen as

$$(2.3) \quad [X, Y] = i\hbar Z$$

$$(2.4) \quad [Y, Z] = i\hbar(2X^3 + XY^2 + Y^2X - 2\mu X)$$

$$(2.5) \quad [Z, X] = i\hbar(2Y^3 + YX^2 + X^2Y - 2\mu Y).$$

The algebra is then defined as $\mathcal{C}_{\hbar, \mu} = \mathbb{C}[X, Y, Z]/I$, where I is the two-sided ideal generated by the above relations. Using the ‘‘Diamond Lemma’’ [Ber78], it was proved that $\mathcal{C}_{\hbar, \mu}$ is a non-trivial algebra for which a basis can be computed [ABH⁺09b]. We shall also consider $\mathcal{C}_{\hbar, \mu}$ to be a $*$ -algebra with $X^* = X$, $Y^* = Y$ and $Z^* = Z$.

In the Poisson algebra, the polynomial C is a Poisson central element of the algebra. It turns out that a non-commutative analogue of C is a central element in $\mathcal{C}_{\hbar, \mu}$. Namely, by setting

$$(2.6) \quad \hat{C} = (X^2 + Y^2 - 2\mu \mathbf{1})^2 + Z^2$$

one computes that $[X, \hat{C}] = [Y, \hat{C}] = [Z, \hat{C}] = 0$. Thus, in analogy with (2.2) it is natural to also impose $\hat{C} = \mathbf{1}$ in $\mathcal{C}_{\hbar, \mu}$. As we shall see, the presentation of the algebra in terms of X , Y and Z , is appropriate when comparing with spherical geometries (and the ‘‘fuzzy sphere’’); there is, however, another choice of basis which naturally makes contact with non-commutative tori. By setting $W = X + iY$ and eliminating $Z = \frac{1}{i\hbar}[X, Y]$, the remaining relations may be written as

$$(2.7) \quad (W^2 W^* + W^* W^2)(1 + \hbar^2) = 4\mu \hbar^2 W + 2(1 - \hbar^2) W W^* W$$

$$(2.8) \quad \frac{1}{4}(W W^* + W^* W - 2\mu \mathbf{1})^2 + \frac{1}{4\hbar^4}(W W^* - W^* W)^2 = \mathbf{1}.$$

Furthermore, by introducing

$$\Lambda = \frac{1}{2\hbar}(WW^* - W^*W) + \frac{i}{2}(WW^* + W^*W - 2\mu\mathbb{1})$$

the above algebra can be presented as

$$(2.9) \quad W\Lambda = q\Lambda W \quad ; \quad W^*\Lambda = \bar{q}\Lambda W^*$$

$$(2.10) \quad W^*\Lambda^* = q\Lambda^*W^* \quad ; \quad W\Lambda^* = \bar{q}\Lambda^*W$$

$$(2.11) \quad \Lambda^*\Lambda = \Lambda\Lambda^* = \mathbb{1}$$

$$(2.12) \quad WW^* = z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}$$

$$(2.13) \quad W^*W = -\bar{z}\Lambda - z\Lambda^* + \mu\mathbb{1},$$

where $q = e^{2\pi i\theta}$, $z = e^{i\pi\theta}/2i \cos \pi\theta$ and θ is related to \hbar via $\hbar = \tan \pi\theta$.

Note that the above relations are quite similar to those of the standard non-commutative torus, generated by two unitary operators. The difference is the deformed unitarity of the operator W . Let us now define the algebra together with the parameter ranges that we shall be interested in.

Definition 2.1. Let $\mu, \theta \in \mathbb{R}$ such that $\mu > 0$ and $|\mu \cos \pi\theta| > 1$. By $A_{\mu,\theta}^0$ we denote the quotient of the (unital) free $*$ -algebra $\mathbb{C}\langle W, W^*, \Lambda, \Lambda^* \rangle$ and the two-sided ideal generated by relations (2.9)–(2.13).

Again, one can make use of the Diamond lemma [Ber78] to explicitly compute a basis of $A_{\mu,\theta}^0$.

Proposition 2.2. A basis for $A_{\mu,\theta}^0$ is given by

$$T_{\vec{m}} = q^{m_1 m_2 / 2} \Lambda^{m_1} W^{m_2}$$

$$S_{\vec{n}} = q^{-n_1 n_2 / 2} \Lambda^{n_1} (W^*)^{n_2}$$

where $\vec{m} = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and $\vec{n} = (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}$. Moreover, it holds that

$$T_{\vec{m}} T_{\vec{n}} = q^{-\vec{m} \times \vec{n} / 2} T_{\vec{m} + \vec{n}}$$

$$S_{\vec{m}} S_{\vec{n}} = q^{\vec{m} \times \vec{n} / 2} S_{\vec{m} + \vec{n}}$$

where $\vec{m} \times \vec{n} = m_1 n_2 - n_1 m_2$.

Proof. To prove that $T_{\vec{m}}$ and $S_{\vec{n}}$ provide a basis for the algebra, we make use of the “Diamond Lemma”, and refer to [Ber78] for details. Thus, relations (2.9)–(2.13) are put into the reduction system

$$S_1 = (W\Lambda, q\Lambda W) \quad S_2 = (W\Lambda^*, \bar{q}\Lambda^*W) \quad S_3 = (W^*\Lambda^*, q\Lambda^*W^*)$$

$$S_4 = (W^*\Lambda, \bar{q}\Lambda W^*) \quad S_5 = (\Lambda\Lambda^*, \mathbb{1}) \quad S_6 = (\Lambda^*\Lambda, \mathbb{1})$$

$$S_7 = (WW^*, z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}) \quad S_8 = (W^*W, -\bar{z}\Lambda - z\Lambda^* + \mu\mathbb{1}),$$

and a compatible ordering is chosen as follows: if two words are of different total order (in $W, W^*, \Lambda, \Lambda^*$) then the one with lower order is smaller than the one with higher order. If two words are of the same order, they are comparable if the orders in $W, W^*, \Lambda, \Lambda^*$ are separately equal. Then the ordering is lexicographic with respect to the alphabet $\Lambda, \Lambda^*, W, W^*$. With this ordering one easily checks that $p_i \geq q_{ij}$, where $S_i = (p_i, \sum_j q_{ij})$. Furthermore, this ordering has the descending chain condition.

There are several ambiguities to be checked in this reduction system. For instance, let us consider $WW^*\Lambda$. One needs to check that it reduces to the same expression if we use S_7 to replace WW^* or S_4 to replace $W^*\Lambda$. One computes

$$\begin{aligned} (z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - W(\bar{q}\Lambda W^*) &= (z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - q\bar{q}\Lambda WW^* \\ &= (z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - \Lambda(z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}) = 0. \end{aligned}$$

The other ambiguities can also be checked to be resolvable. Hence, by the Diamond Lemma, a basis for the algebra is provided by the irreducible words. Denoting $(\Lambda^*)^n = \Lambda^{-n}$ the irreducible words are given by $T_{\vec{m}}$ and $S_{\vec{n}}$. To prove the product formulas, one simply uses the relations to reorder the expressions. \square

2.1. Scaling limits. Let us now show that the algebras defined above have two formal scaling limits reproducing the fuzzy sphere and the standard non-commutative torus, as well as their finite dimensional representations.

For the sphere, one introduces $\tilde{X} = X/\varepsilon$, $\tilde{Y} = Y\varepsilon$, $\tilde{Z} = Z/\varepsilon$ as well as $k = \hbar/\varepsilon$. In terms of the rescaled variables, relations (2.3)–(2.5) become

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= ik\tilde{Z} \\ [\tilde{Y}, \tilde{Z}] &= -2ik\mu\tilde{X} + ik\varepsilon^2\Phi(\tilde{X}, \tilde{Y}) \\ [\tilde{Z}, \tilde{X}] &= -2ik\mu\tilde{Y} + ik\varepsilon^2\Phi(\tilde{Y}, \tilde{X}), \end{aligned}$$

where $\Phi(X, Y) = 2X^3 + XY^2 + Y^2X$. For $-1 < \mu < 0$ one finds an algebra isomorphic to $\mathfrak{su}(2)$ as $\varepsilon \rightarrow 0$. In [ABH⁺09b], the non-zero matrix elements of $W = X + iY$, in an N -dimensional irreducible representation, were found to be

$$W_{l,l+1} = \sqrt{\frac{2 \sin(\pi l \theta) \sin \pi(n-l)\theta}{\cos \pi \theta}}$$

for $l = 1, \dots, N-1$. Setting $\tilde{W} = \tilde{X} + i\tilde{Y}$ and $k = \tan \tilde{\theta}$, one finds that $\theta = \varepsilon \tilde{\theta}$ for small ε , and

$$\tilde{W}_{l,l+1} \approx \frac{1}{\varepsilon} \sqrt{2\pi^2 \varepsilon^2 l(n-l) \tilde{\theta}^2} \rightarrow \sqrt{2\pi \tilde{\theta} \sqrt{l(n-l)}},$$

which reproduces a standard representation of $\mathfrak{su}(2)$.

To obtain the non-commutative torus, one considers relations (2.9)–(2.13) and introduces $\tilde{W} = \varepsilon W$ and sets $\mu = 1/\varepsilon^2$. Relations (2.12) and (2.13) then become

$$\begin{aligned} \tilde{W}\tilde{W}^* &= \varepsilon^2(z\Lambda + \bar{z}\Lambda^*) + \mathbb{1} \\ \tilde{W}^*\tilde{W} &= \varepsilon^2(-z\Lambda - \bar{z}\Lambda^*) + \mathbb{1} \end{aligned}$$

which reduces to the fact that \tilde{W} is unitary as $\varepsilon \rightarrow 0$. Clearly, equations (2.9)–(2.11) are invariant under this rescaling. The corresponding non-zero matrix elements are

$$\begin{aligned} W_{N,1} &= \sqrt{\mu + \frac{1}{\cos \theta}} \\ W_{l,l+1} &= \sqrt{\mu + \frac{\cos 2\pi l \theta}{\cos \theta}} \end{aligned}$$

for $l = 1, \dots, N - 1$, which implies that

$$\begin{aligned}\tilde{W}_{N,1} &= \varepsilon \sqrt{\frac{1}{\varepsilon^2} + \frac{1}{\cos \theta}} \rightarrow 1 \\ \tilde{W}_{l,l+1} &= \varepsilon \sqrt{\frac{1}{\varepsilon^2} + \frac{\cos 2\pi l \theta}{\cos \theta}} \rightarrow 1\end{aligned}$$

as $\varepsilon \rightarrow 0$. Even without any rescaling, it holds that Λ is a matrix with the N roots of unity on the diagonal. This reproduces the finite dimensional representations of the non-commutative torus.

3. RELATION TO THE STANDARD NON-COMMUTATIVE TORUS

Let U, V be the generators of the non-commutative torus C_θ [Con80, Con94, CR87]; i.e. the universal C^* -algebra generated by the relations

$$\begin{aligned}VU &= e^{i2\pi\theta}UV \\ U^*U &= UU^* = \mathbb{1} \\ V^*V &= VV^* = \mathbb{1}.\end{aligned}$$

In what follows, we will show that one can map $A_{\mu,\theta}^0$ into C_θ and use the induced norm to complete $A_{\mu,\theta}^0$ to a C^* -algebra isomorphic to C_θ . Let us start by proving a result about the spectrum of a particular element in C_θ , that is used to construct a $*$ -homomorphism from $A_{\mu,\theta}^0$ to C_θ .

Lemma 3.1. *If $|\mu \cos \pi\theta| > 1$ and $\mu > 0$ then the element $\mu\mathbb{1} + ze^{i\pi\varphi}U + \bar{z}e^{-i\pi\varphi}U^*$, with $z = e^{i\pi\theta}/2i \cos \pi\theta$, is positive and invertible in C_θ for all $\varphi \in \mathbb{R}$.*

Proof. The element is clearly hermitian and let us write

$$\mu\mathbb{1} + ze^{i\pi\varphi}U + \bar{z}e^{-i\pi\varphi}U^* \equiv \mu\mathbb{1} - B.$$

To study the spectrum, we consider the invertibility of the element $(\mu - \lambda)\mathbb{1} - B$ for different λ . It is a standard fact that this element is invertible if $\frac{1}{|\mu - \lambda|} \|B\| < 1$. One computes

$$\frac{1}{|\mu - \lambda|} \|B\| = \frac{1}{2|(\mu - \lambda) \cos \pi\theta|} \|e^{i\pi\varphi}U + e^{i\pi\varphi}U^*\| \leq \frac{1}{|(\mu - \lambda) \cos \pi\theta|},$$

which is less than one if $|(\mu - \lambda) \cos \pi\theta| > 1$. Since $|\mu \cos \pi\theta| > 1$ by assumption (and $\mu > 0$), it follows that $\frac{1}{|\mu - \lambda|} \|B\| < 1$ for all $\lambda \leq 0$. Hence, $\mu\mathbb{1} - B$ is invertible and the spectrum is contained in $(0, \infty)$. \square

Thus, it follows from Lemma 3.1 that if $\mu > 0$ is chosen such that $|\mu \cos \pi\theta| > 1$ then both $\sqrt{\mu\mathbb{1} + zU + \bar{z}U^*}$ and its inverse exist in C_θ .

Proposition 3.2. *The map ϕ , defined by*

$$\begin{aligned}\phi(W) &= \left(\sqrt{\mu\mathbb{1} + zU + \bar{z}U^*} \right) V \\ \phi(\Lambda) &= U,\end{aligned}$$

induces an injective $$ -homomorphism from $A_{\mu,\theta}^0$ to C_θ .*

Proof. First of all, one has to check that the map is well defined, i.e. that it respects the relations in $A_{\mu,\theta}^0$; for instance, denoting $R = \mu\mathbb{1} + zU + \bar{z}U^*$, one computes

$$\begin{aligned}\phi(WW^* - z\Lambda - \bar{z}\Lambda^* - \mu\mathbb{1}) &= \sqrt{R}VV^*\sqrt{R} - zU - \bar{z}U^* - \mu\mathbb{1} \\ &= R - zU - \bar{z}U^* - \mu\mathbb{1} = 0,\end{aligned}$$

by using the relations in C_θ . The remaining relations are checked in a similar way. Now, let us prove that ϕ is injective. It follows from Proposition 2.2 that an arbitrary element $a \in A_{\mu,\theta}^0$ can be written as

$$a = \sum_{\vec{m} \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}} a_{\vec{m}} \Lambda^{m_1} W^{m_2} + \sum_{\vec{n} \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}} b_{\vec{n}} \Lambda^{n_1} (W^*)^{n_2}$$

(where all but a finite number of coefficients are zero) which implies that

$$\phi(a) = \sum_{\vec{m}} a_{\vec{m}} U^{m_1} (\sqrt{R}V)^{m_2} + \sum_{\vec{n}} b_{\vec{n}} U^{n_1} (V^* \sqrt{R})^{n_2}.$$

Note that Lemma 3.1 implies that $R(q^k) = \mu\mathbb{1} + zq^kU + \bar{z}q^kU^*$ is positive and invertible for all $k \in \mathbb{Z}$, which in particular implies that $(\sqrt{R})V = V\sqrt{R(\bar{q})}$ and $V^*\sqrt{R} = (\sqrt{R(\bar{q})})V^*$. Thus, one concludes that

$$\phi(a) = \sum_{\vec{m}} a_{\vec{m}} \left(\prod_{k=0}^{m_2-1} \sqrt{R(\bar{q}^k)} \right) U^{m_1} V^{m_2} + \sum_{\vec{n}} b_{\vec{n}} \left(\prod_{k=1}^{n_2} \sqrt{R(\bar{q}^k)} \right) U^{n_1} (V^*)^{n_2}.$$

Since elements of the form

$$\sum_{\vec{m} \in \mathbb{Z} \times \mathbb{Z}} c_{\vec{m}} U^{m_1} V^{m_2}$$

form a basis of a dense subset of C_θ , it follows that if $\phi(a) = 0$ then one must have $a_{\vec{m}} = b_{\vec{n}} = 0$ for all \vec{m} and \vec{n} . Hence, $a = 0$, which proves that ϕ is injective. \square

Since ϕ is injective, one can define a C^* -norm on $A_{\mu,\theta}^0$ by setting $\|a\| = \|\phi(a)\|$ for all $a \in A_{\mu,\theta}^0$, and by $A_{\mu,\theta}$ we denote the completion of $A_{\mu,\theta}^0$ in this norm. Moreover, ϕ can be extended to $A_{\mu,\theta}$ by continuity, and (by a slight abuse of notation) we shall also denote the extended map by ϕ .

Proposition 3.3. *The map $\phi : A_{\mu,\theta} \rightarrow C_\theta$ is an isomorphism of C^* -algebras.*

Proof. As in Lemma 3.1, one can show that $\mu\mathbb{1} + z\Lambda + \bar{z}\Lambda^*$ is positive and invertible in $A_{\mu,\theta}$. Hence, one constructs the inverse of ϕ by setting

$$\begin{aligned}\phi^{-1}(V) &= \frac{1}{\sqrt{\mu\mathbb{1} + z\Lambda + \bar{z}\Lambda^*}} W \\ \phi^{-1}(U) &= \Lambda.\end{aligned}$$

(which is easily shown to be a well defined map) and extending it as a $*$ -homomorphism through continuity. \square

3.1. Projective modules. In [ABH⁺09b] all finite-dimensional hermitian $*$ -representations of $\mathcal{C}_{\hbar,\mu}$ were constructed and classified. It was found that, in the case of algebras related to tori, the parameter θ has to be a rational number for finite dimensional representations to exist; which is in the same spirit as for C_θ . For the sake of comparison, let us see how the standard projective modules of the non-commutative torus can be presented for $A_{\mu,\theta}$.

Let $\xi_{m,n}$ be the vector space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}_n)$, i.e. the space of Schwartz functions in one real variable x and one discrete variable $k \in \mathbb{Z}_n$. By defining

$$(3.1) \quad (\phi W)(x, k) = W(x, k) \phi(x - \varepsilon, k - 1)$$

$$(3.2) \quad (\phi W^*)(x, k) = W(x + \varepsilon, k + 1) \phi(x + \varepsilon, k + 1)$$

$$(3.3) \quad (\phi \Lambda)(x, k) = e^{2\pi i(x - mk/n)} \phi(x, k)$$

$$(3.4) \quad (\phi \Lambda^*)(x, k) = e^{-2\pi i(x - mk/n)} \phi(x, k),$$

where $\varepsilon = (m + n\theta)/n$ and

$$(3.5) \quad W(k, x) = \left(\mu + \frac{\sin(2\pi(x - mk/n) - \pi\theta)}{\cos \pi\theta} \right)^{\frac{1}{2}},$$

one can check that $\xi_{m,n}$ becomes a right $A_{\mu,\theta}^0$ module.

The standard derivations on C_θ , defined by

$$\begin{aligned} \partial_1 U &= iU & \partial_2 U &= 0 \\ \partial_1 V &= 0 & \partial_2 V &= iV, \end{aligned}$$

and extended to the smooth part of C_θ , can be pulled back to the smooth part of $A_{\mu,\theta}$ (defined as the inverse image of the smooth part of C_θ) giving

$$\begin{aligned} \partial_1 \Lambda &= i\Lambda & \partial_2 \Lambda &= 0 \\ \partial_1 W &= i(z\Lambda - \bar{z}\Lambda^*)(\mu\mathbb{1} + z\Lambda + \bar{z}\Lambda^*)^{-1}W \\ \partial_2 W &= iW. \end{aligned}$$

Furthermore, a connection may be defined on the above modules in a standard manner. Namely, the linear operators $\nabla_1, \nabla_2 : \xi_{m,n} \rightarrow \xi_{m,n}$, given as

$$\begin{aligned} (\nabla_1 \phi)(x, k) &= \frac{1}{2\pi} \frac{d\phi}{dx}(x, k) \\ (\nabla_2 \phi)(x, k) &= \frac{i}{\varepsilon} x \phi(x, k), \end{aligned}$$

define a connection on $\xi_{m,n}$, i.e. they fulfill

$$\nabla_i(\phi \cdot a) = (\nabla_i \phi) \cdot a + \phi \cdot (\partial_i a)$$

for $i = 1, 2$ and a in the smooth part of $A_{\mu,\theta}$. One easily computes that

$$[\nabla_1, \nabla_2] = \frac{i}{2\pi\varepsilon} \mathbb{1},$$

i.e. the connection has constant curvature.

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